Low-Complexity Integer Forcing for Block Fading MIMO Multiple-Access Channels

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***Abstract*— Integer forcing is an alternative approach to conventional linear receivers in multiple-antenna systems. In the integer-forcing scheme, the receivers try to extract integer combinations of messages from the received matrix before recovering each message individually. Recently, the integer-forcing approach was generalized to a block fading scenario. Among the decoding methods described, the one which achieves higher rates has the drawback that no efficient algorithm to find the best choice of integer linear combinations is known. In this paper, we propose a sub-optimal scheme to find those combinations with a lower complexity. Simulation results show that the proposed method outperforms other existing low-complexity methods.**

***Keywords*— Integer-forcing, block fading, MIMO channels**

I. Introduction

Integer-forcing (IF) receivers are an alternative to traditional methods of equalization, such as zero-forcing (ZF) and minimum-mean-squared-error (MMSE) equalization [1] for multiple-input and multiple-output (MIMO) channels. The IF approach follows from the compute-and-forward framework [2], [7] for relay networks, where the receivers attempt to extract integer linear combinations of the transmitted messages from the received signals, before recovering the messages themselves.

Although joint maximum likelihood (ML) receivers achieve the best performance of all methods, by searching all possible codewords [3], their complexity is prohibitively high, since it increases exponentially with the number of users. On the other hand, IF receivers have a much lower complexity and can approach the performance of ML in many situations [1].

The main results about IF consider static fading, where all symbols of a codeword are subject to the same channel fading. Note that in a practical situation, where a powerful code with large block-length is used, it may not be realistic to assume that all symbols of the codeword are subject to the same channel fading. Therefore, channels that allow block fading [4], where the channel fading can vary during the transmission of a codeword, seem to be a more realistic model.

In a recent work, El Bakoury and Nazer [5] generalize the IF approach to a block fading scenario. They described two decoding methods for block fading, which are called AM (arithmetic mean) and GM (geometric mean) decoding. The AM-decoding method approximates the effective noise among all block as static-fading effective noise. On the other hand, the GM decoding method attempts to extract the maximum amount of information from each block independently.

Both decoding methods can achieve higher rates by optimizing the choice of the linear combinations, with the GM method always outperforming the AM method for the same combination [5]. However, finding the best choice of linear linear combinations for the GM method is too complex [6], since no efficient algorithm is known [5].

Our contribution is a scheme to find a sub-optimal choice for the integer linear combinations for the GM-decoding method. The proposed scheme is based on a choice between two possible candidates, one of them being the optimal solution for the AM method. Simulation results show that the proposed method can achieve higher rates than optimal AM while being less complex than the optimal GM and ML.

In the remainder of this paper, we begin describing the system model in Section II and then, in Section III a brief review about integer forcing for static fading as well as block fading. In Section IV we detail the proposed scheme. Simulation results are shown in Section V and lastly, we conclude in Section VI.

II. System Model

We denote row-vectors as lowercase-bold letters (e.g. $\mathbf{x}$) and matrices as uppercase bold letters (e.g. $\mathbf{X}$). Let the matrices $\mathbf{X}^{\mathrm{T}}$ and $\mathbf{X}^{-1}$ be the transpose and inverse of $\mathbf{X}$, respectively. Let $\mathbb{Z}_p$ be the field of integers modulo p, where p is a prime.

We are interested in the scenario with N_T single-antenna transmitters and one N_R-antenna receiver. The information data of the ℓ-th transmitter, $\mathbf{w}_\ell \in \mathbb{Z}_p^k$, is coded/modulated into the codeword $\mathbf{x}_\ell = \begin{bmatrix}\mathbf{x}_\ell[1] & \dots & \mathbf{x}_\ell[n]\end{bmatrix} \in \mathbb{R}^n$. The message rate is defined as $R_{\mathrm{mes}} \triangleq \log_2(p)\frac{k}{n}$. The vector transmitted must satisfy the power constraint, i.e., $\frac{1}{n}\sum_i \left|\mathbf{x}_\ell[i]\right|^2 \leq P$. Let $\mathbf{X} \in \mathbb{R}^{N_T \times n}$ be the matrix such that the ℓ-th row of $\mathbf{X}$ is $\mathbf{x}_\ell$, i.e., $\mathbf{X} = \begin{bmatrix}\mathbf{x}_1^{\mathrm{T}} & \cdots & \mathbf{x}_{N_T}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$.

Let $\mathbf{Y} \in \mathbb{R}^{N_R \times n}$ be the matrix such that the j-th row is the signal received by the j-th antenna at the receiver, $j = 1, \dots, N_R$. Let N be a strictly positive integer such that N divides n. In the block fading scenario, the received matrix can be written as $\mathbf{Y} = \begin{bmatrix}\mathbf{Y}^{(1)} & \cdots & \mathbf{Y}^{(N)}\end{bmatrix}$ where

$$\mathbf{Y}^{(i)} = \mathbf{H}^{(i)}\mathbf{X}^{(i)} + \mathbf{Z}^{(i)} \tag{1}$$

$\mathbf{H}^{(i)} \in \mathbb{R}^{N_R \times N_T}$ is the channel matrix for the i-th block, $\mathbf{X}^{(i)}$ is such that $\mathbf{X} = \begin{bmatrix}\mathbf{X}^{(1)} & \cdots & \mathbf{X}^{(N)}\end{bmatrix}$ and $\mathbf{Z}^{(i)}$ is Gaussian noise with i.i.d. entries of zero mean and variance N_0.

For convenience, the signal-to-noise ratio is defined as

$$\mathrm{SNR} \triangleq \frac{P}{N_0}. \tag{2}$$

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The decoder $\mathcal{D}(\mathbf{Y}) = (\hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_{N_T})$ tries to recover all messages. We say that an outage occurs when $\hat{\mathbf{w}}_\ell \neq \mathbf{w}_\ell$ for any ℓ and the outage probability for a given R_{mes} is defined as $p_{out}(R_{\text{mes}}) = \mathcal{P}[(\hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_{N_T}) \neq (\mathbf{w}_1, \ldots, \mathbf{w}_{N_T})]$. For a given outage probability ρ we define the outage rate $R(\rho) = \sup\{R_{\text{mes}} : p_{out}(R_{\text{mes}}) < \rho\}$.

III. Integer Forcing

In the integer-forcing approach, the decoder is interested in recovering N_T linearly independent integer linear combinations of messages $\mathbf{U} = \mathbf{A}\mathbf{W} \mod p$ [1] where $\mathbf{W} \in \mathbb{Z}_p^{N_T \times n}$ is a matrix where each line correspond to the message $\mathbf{w}_\ell$ and $\mathbf{A} \in \mathbb{Z}^{N_T \times N_T}$ is the matrix with the integer coefficients of the linear combinations.

Integer forcing considers that a lattice codebook is used by all transmitters, i.e., all codewords are elements of the same lattice [2]. Note that any integer linear combinations of elements of a lattice is also in the lattice [2]. Moreover, there must be a linear map φ from codewords to message space such that $\varphi(\mathbf{X}) = \mathbf{W}$. This means that if $\mathbf{V} = \mathbf{A}\mathbf{X}$ can be decoded from the received matrix $\mathbf{Y}$, then matrix $\mathbf{U}$ can be recovered as $\varphi(\mathbf{V}) = \mathbf{U}$ [1].

We first explain integer forcing for static fading, where $N = 1$. Then we show the necessary modifications for block fading, where $N \geq 2$.

A. Static Fading

Consider the channel in (1) (the superscript will be omitted). The receiver applies the equalization matrix $\mathbf{B} \in \mathbb{R}^{N_T \times N_R}$ to create an effective channel output

$$\begin{aligned} \mathbf{Y}_{\text{eff}} &= \mathbf{B}\mathbf{Y} = \mathbf{B}\mathbf{H}\mathbf{X} + \mathbf{B}\mathbf{Z} \\ &= \mathbf{A}\mathbf{X} + \mathbf{N}_{\text{eff}} \end{aligned} \tag{3}$$

where

$$\mathbf{N}_{\text{eff}} = (\mathbf{B}\mathbf{H} - \mathbf{A})\mathbf{X} + \mathbf{B}\mathbf{Z} \tag{4}$$

is the effective noise [1].

The variance of the m-th row of the effective noise is

$$\sigma^2_{\text{eff},m} = N_0 \left(\left\| \mathbf{b}_m \mathbf{H} - \mathbf{a}_m \right\|^2 \text{SNR} + \left\| \mathbf{b}_m \right\|^2 \right) \tag{5}$$

where $\mathbf{b}_m$ is the m-th row of the equalization matrix $\mathbf{B}$ and $\mathbf{a}_m$ is the m-th row of matrix $\mathbf{A}$.

It is intuitive and can be shown that minimizing the effective noise variance maximizes the achievable rate [2]. The optimal equalization matrix, for a given $\mathbf{A}$, can be found using MMSE estimation [1] and is given by

$$\mathbf{B}_{\text{opt}} = \text{SNR}\mathbf{A}\mathbf{H}^{\text{T}}(\mathbf{I} + \text{SNR}\mathbf{H}\mathbf{H}^{\text{T}})^{-1} \tag{6}$$

From now on we assume that the optimal equalization matrix is used, i.e., $\mathbf{B} = \mathbf{B}_{\text{opt}}$. In this case, the variance of the effective noise can be written as [7]

$$\sigma^2_{\text{eff},m} = N_0(\mathbf{a}_m \mathbf{M} \mathbf{a}_m^{\text{T}}) \tag{7}$$

where

$$\mathbf{M} = (\text{SNR}^{-1}\mathbf{I} + \mathbf{H}^{\text{T}}\mathbf{H})^{-1}. \tag{8}$$

Theorem 1: [1], [2] For any $\epsilon > 0$ and n, p sufficiently large, there is a scheme of encoders and decoders such that it is possible to recover m integer linear combinations from the channel (3) with outage probability at most ϵ for any choice of $\mathbf{A}$ if

$$R_{\text{mes}} < R_{\text{comp}}(\mathbf{A}) = \min_m R_{\text{comp}}(\mathbf{a}_m) \tag{9}$$

where

$$\begin{aligned} R_{\text{comp}}(\mathbf{a}_m) &= \frac{1}{2} \log^+ \left(\frac{P}{\sigma^2_{\text{eff},m}} \right) \\ &= \frac{1}{2} \log^+ \left(\frac{\text{SNR}}{\mathbf{a}_m \mathbf{M} \mathbf{a}_m^{\text{T}}} \right) \end{aligned} \tag{10}$$

and $\log^+(\cdot) = \max(\log(\cdot), 0)$.

To choose the optimal matrix $\mathbf{A}$ we have to find a basis, with minimal norm, of the lattice generated by $\mathbf{M}$ [1]. It is believed that finding such basis is a NP-Hard problem [6], however there are suboptimal algorithms that can find an approximation in polynomial time, for example the LLL algorithm[1] [8].

Note that if $\mathbf{A} = \mathbf{I}$ is chosen then the scheme reduces to MMSE equalization [1].

B. Block Fading

Assume that the channel equation is defined by (1). It is possible to equalize each block independently using MMSE estimation. The equalization matrix for each block, for a given $\mathbf{A}$, is

$$\mathbf{B}^{(i)} = \text{SNR}\mathbf{A}(\mathbf{H}^{(i)})^{\text{T}} \left(\mathbf{I} + \text{SNR}\mathbf{H}^{(i)}(\mathbf{H}^{(i)})^{\text{T}} \right)^{-1}. \tag{11}$$

The effective channel output is given by

$$\begin{aligned} \mathbf{Y}^{(i)}_{\text{eff}} &= \mathbf{B}^{(i)}\mathbf{Y}^{(i)} = \mathbf{B}^{(i)}\mathbf{H}^{(i)}\mathbf{X}^{(i)} + \mathbf{B}^{(i)}\mathbf{Z}^{(i)} \\ &= \mathbf{A}\mathbf{X}^{(i)} + \mathbf{N}^{(i)}_{\text{eff}}. \end{aligned} \tag{12}$$

Note that matrix $\mathbf{A}$ must not change in each block [5]. That is because $\mathbf{v}_m = \mathbf{a}_m\mathbf{X}$ should be an element of the lattice.

The variance of the effective noise, in each block, is

$$\begin{aligned} {\sigma^2_{\text{eff},m}}^{(i)} &= N_0 \left(\left\| \mathbf{b}^{(i)}_m \mathbf{H}^{(i)} - \mathbf{a} \right\|^2 \text{SNR} + \left\| \mathbf{b}^{(i)}_m \right\|^2 \right) \\ &= N_0 \left(\mathbf{a}_m \mathbf{M}^{(i)} \mathbf{a}_m^{\text{T}} \right) \end{aligned} \tag{13}$$

where

$$\mathbf{M}^{(i)} = \left(\text{SNR}^{-1}\mathbf{I} + (\mathbf{H}^{(i)})^{\text{T}}\mathbf{H}^{(i)} \right)^{-1} \tag{14}$$

and the optimal equalization matrix is used.

It is shown in [5] that Theorem 1 is still valid in this case, however the expression for $R_{\text{comp}}(\mathbf{a}_m)$ changes depending on the decoding method. The methods proposed in [5] are the AM (arithmetic mean) and the GM (geometric mean) methods discussed below.

[1] For the 2×2 channel, it is possible to find the optimal basis in polynomial time [6]. For large networks the use of LLL algorithm is recommended.

1) AM Method: This decoding method considers that all blocks are subject to the same variance of the effective noise [5]. This variance is given by the *arithmetic mean* of the variance of each block, i.e.,

$$\sigma^2_{\mathrm{AM},m} = \frac{1}{N}\sum_{i=1}^{N} {\sigma^2_{\mathrm{eff},m}}^{(i)}. \tag{15}$$

The computation rate that can be achieved is [5]

$$\begin{aligned} R_{\mathrm{comp}}(\mathbf{a}_m) &= \frac{1}{2}\log^+\left(\frac{P}{\sigma^2_{\mathrm{AM},m}}\right) \\ &= \frac{1}{2}\log^+\left(\frac{\mathrm{SNR}}{\mathbf{a}_m\mathbf{M}_{\mathrm{AM}}\mathbf{a}_m^{\mathrm{T}}}\right) \end{aligned} \tag{16}$$

where

$$\mathbf{M}_{\mathrm{AM}} = \frac{1}{N}\sum_i \mathbf{M}^{(i)} \tag{17}$$

and $\mathbf{M}^{(i)}$ is the same as (14).

2) GM Method: In this method, the decoder considers the effective-noise variance of each block independently. The computation rate that this method can achieve is [5]

$$\begin{aligned} R_{\mathrm{comp}}(\mathbf{a}_m) &= \min_m \frac{1}{2N}\sum_i \log^+\left(\frac{P}{{\sigma^2_{\mathrm{eff},m}}^{(i)}}\right) \\ &= \min_m \frac{1}{2N}\sum_i \log^+\left(\frac{\mathrm{SNR}}{\mathbf{a}_m\mathbf{M}^{(i)}\mathbf{a}_m^{\mathrm{T}}}\right) \\ &= \min_m \frac{1}{2}\log^+\left(\frac{P}{\sigma^2_{\mathrm{GM},m}}\right) \end{aligned} \tag{18}$$

where $\mathbf{M}^{(i)}$ is defined in (14) and

$$\sigma^2_{\mathrm{GM},m} = \left(\prod_i {\sigma^2_{\mathrm{eff},m}}^{(i)}\right)^{\frac{1}{N}}. \tag{19}$$

Note that (19) is the *geometric mean* of the variance of the effective noise in each block.

For the same matrix $\mathbf{A}$, the GM decoder can achieve a higher rate than the AM decoder since $\sigma^2_{\mathrm{GM}} \leq \sigma^2_{\mathrm{AM}}$ due to AM-GM inequality [9].

C. Choosing matrix $\mathbf{A}$

For the AM decoder, finding the optimal $\mathbf{A}$ can be done in the same way as for the static case, i.e, the LLL algorithm (or similar) can be used to find a basis, with minimal norm, of the lattice generated by $\mathbf{M}_{\mathrm{AM}}$ (instead of $\mathbf{M}$) in polynomial time [5]. When the optimal $\mathbf{A}$ for the AM decoding method is used, we refer to the scheme as the AM-IF method.

Of course, the identity matrix can also be used in the AM decoder. In this case, we call this scheme the AM-MMSE method.

Similarly, for the GM decoder, if the optimal matrix is used then we refer to this scheme as the GM-IF method and if the identity matrix is used we call it the GM-MMSE method.

Note that there is no known efficient method to find the optimal matrix in the GM decoder due to the complexity of the optimization problem in (18) [5].

IV. PROPOSED SCHEME

As an alternative to find the optimal matrix in the GM decoder, we propose a sub-optimal choice for the matrix $\mathbf{A}$. In our proposed scheme the matrix $\mathbf{A}$ is chosen as either the optimal matrix in the AM method or the identity matrix.

Note that, if the matrix chosen is the $\mathbf{A}_{\mathrm{AM}}$ (the optimal in the AM method) the proposed scheme can achieve higher rates than the AM method. On the other hand, if the matrix chosen is the identity then the proposed scheme is the same as the GM-MMSE, and therefore achieves the same rate.

Below, in Algorithm 1, we summarize the procedure of the proposed method.

Algorithm 1 Choosing matrix $\mathbf{A}$ for the proposed method

Input: $\mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(N)}$, SNR
1: Calculate $\mathbf{M}_{\mathrm{AM}}$ as (17);
2: Find matrix $\mathbf{A}_{\mathrm{AM}}$ using LLL algorithm (or similar)
3: Calculate $R_{\mathrm{comp}}(\mathbf{A}_{\mathrm{AM}})$ using (18).
4: Set $\mathbf{A}_{\mathrm{MMSE}} = \mathbf{I}$;
5: Calculate $R_{\mathrm{comp}}(\mathbf{A}_{\mathrm{MMSE}})$ using (18).
6: **if** $R_{\mathrm{comp}}(\mathbf{A}_{\mathrm{AM}}) > R_{\mathrm{comp}}(\mathbf{A}_{\mathrm{MMSE}})$ **then**
7: Set $\mathbf{A} = \mathbf{A}_{\mathrm{AM}}$
8: **else**
9: Set $\mathbf{A} = \mathbf{A}_{\mathrm{MMSE}}$
10: **end if**
Output: Matrix $\mathbf{A}$

The complexity of the proposed method is dominated by that of finding the optimal matrix in the AM method (step 2 the algorithm), which can be done in polynomial time with the LLL algorithm. Therefore the complexity is the same as the AM-IF. The proposed method is more complex than the GM-MMSE, but can potentially achieve higher rates as show in the next section.

V. SIMULATION RESULTS

In this section we show some simulation results comparing the proposed method to AM-IF and GM-MMSE in a real-valued channel. In our simulations, we specify the outage probability $\rho = 0.01$. In each scenario 10000 channel realizations was made. In each realization the channel coefficients are draw randomly by Gaussian distribution with zero mean and unit variance.

In Fig. 1 we show the comparison between the methods in a 2×2 real channel with 2 blocks. We include the ML decoder as an upper bound. For comparison, we also include the outage rate for $N = 1$ for comparison as well as the GM-IF method. The latter was obtained by an exhaustive search for all integer vector such that the magnitude of each entry does not exceed 10.

As expected, the GM-IF has the higher outage rate between the methods. The proposed method is outperformed by approximately 3 dB in high SNR in comparison with GM-IF. It is interesting to note that the outage rate for the proposed scheme is strictly higher than the maximum between the AM-IF and GM-MMSE.

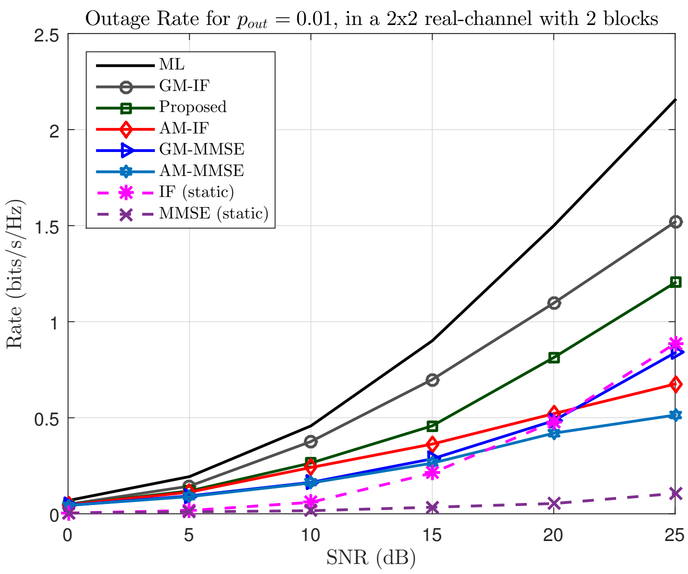

Fig. 1. Comparison of the outage rate for 2 blocks. For comparison, the static case ($N = 1$) is also included.

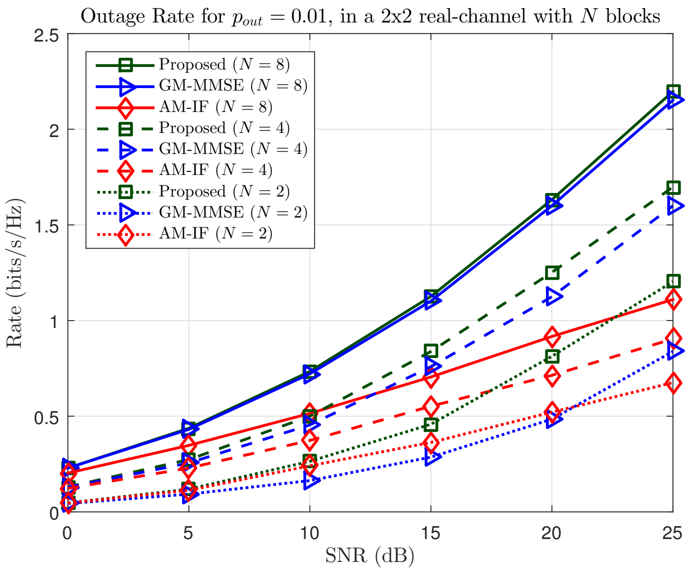

Fig. 2. Outage rate for $N_T = N_R = 2$ when the number of block varies.

In Fig. 2 we plot the outage rate as the number of blocks varies in a 2×2 real channel. In this and following figures we do not include the GM-IF method, since it has higher rates than any other method. We also do not include the AM-MMSE method because its rates is always inferior than both AM-IF and GM-MMSE. Note that, as the number of blocks increases the GM-MMSE method can achieve higher rates than the AM-IF. On the other hand, the proposed scheme achieves higher rates than both GM-MMSE and AM-IF, as expected.

In Fig. 3 we show the comparison between the methods for 2 blocks in a real channel, varying the number of users (the number of antennas at the receiver always matches with the number of users). In this case, the situation is reversed: AM-IF achieves higher rates than GM-IF. Again, the proposed method achieves rates higher than those of both methods.

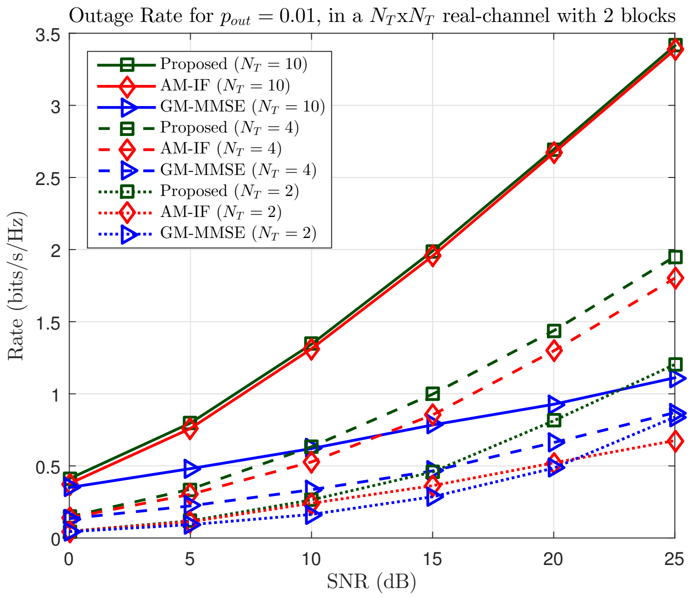

Fig. 3. Outage rate for $N = 2$ when the number of users varies (and considering that $N_T = N_R$).

TABLE I

OUTAGE PROBABILITY OVER A 2×2 CHANNEL WITH $N = 2$ BLOCKS AND SNR = 9.3 DB.

		AM-IF	
		Outage	Success
GM-MMSE	Outage	7.25%	4.64%
	Success	4.65%	83.46%

	Outage	Success
Proposed method	5.69%	94.31%

We can better understand the superior performance of the proposed method by noticing that it can succeed even when both AM-IF and GM-MMSE fail. Table I shows the outage probability for AM-IF and GM-MMSE over a 2×2 real-valued channel, with $N = 2$ blocks, $R_{\text{mes}} = 0.5$ and SNR $= 9.3$ dB. This value of SNR was chosen so that both methods have the same outage probability, about 11.9%. In principle, one would expect the proposed method to fail whenever both AM-IF and GM-MMSE fail simultaneously, which occurs with probability 7.25%. However, the outage probability of the proposed method is even smaller, only 5.69%. This can be explained by the fact that, for the same matrix $\mathbf{A}$, the GM decoder has always better performance than the AM decoder. Thus, the proposed method can exploit the same improved matrix $\mathbf{A}$ as the AM-IF decoder, but without being limited by its achievable rates.

Lastly, in Fig. 4 we show the frame-error rate (FER) in a scenario where a root-LDPC code [10] is used with a BPSK modulation. The root-LDPC code is a rate $1/2$, regular $(3, 6)$-code of length 2000 constructed using a PEG-based technique [11]. The LLR (log-likelihood ratio) used in the root-LDPC decoder was computed by approximating the effective noise distribution with a Gaussian distribution. We also plot the

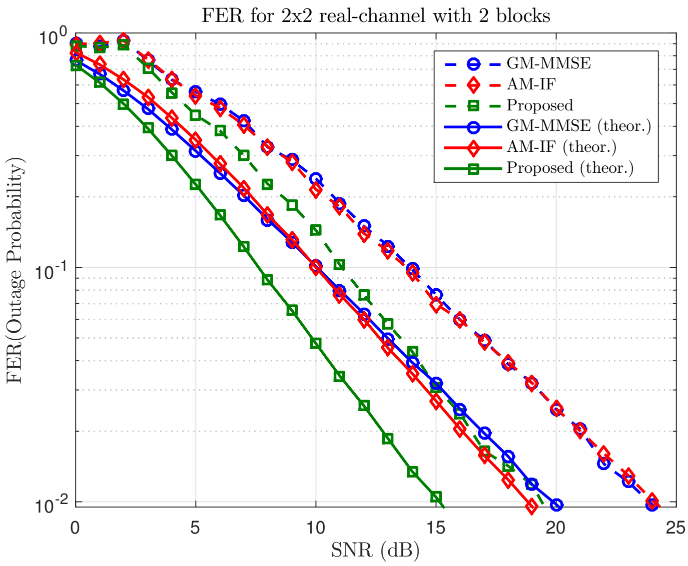

Fig. 4. FER (and outage probability) in a 2×2 real channel with a LDPC code.

theoretical outage probability for comparison. It can be seen that the proposed method can achieve better performance than the AM-IF and GM-MMSE. Moreover, the proposed method has a loss of only about 4 dB for an FER of 1% in comparison to its theoretical outage probability. This loss is the same as that of the GM-MMSE and lower than that of the AM-IF (5 dB).

It is important to note that the highest code rate for a root-LDPC code is $1/N$, where N is the number of blocks [10]. Therefore, in order to improve the R_{mes}, it would be necessary spectral efficiency lattice-codes designed for block fading, which to the best of our knowledge is still an open problem.

VI. Conclusions

In this paper, we proposed a new sub-optimal scheme for decoding in a block fading scenario using a GM-type decoder. For a large number of users, using GM-MMSE does not seem advantageous since AM-IF outperforms it. However if there is a large number of blocks, the GM-MMSE can achieve higher rates than the AM-IF. The proposed method uses the best matrix $\mathbf{A}$ between those two methods in a GM-type decoder. As shown by simulation results, the proposed method achieves computational rates strictly higher than both GM-MMSE and AM-IF, regardless of the number of blocks and users. Moreover, it achieves higher rates than the maximum between GM-MMSE and AM-IF, while being only sightly more complex than the GM-MMSE.

It was also shown that when a root-LDPC code is used, the proposed method can outperform GM-MMSE and AM-IF, as well. These results imply that much of the performance gains promised by integer forcing can be realized in practice with low complexity, even in a block fading channel.

Acknowledgments

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