Closed-Form Approximations to the High-order Statistics of the κ-μ Extreme Fading

Érick Mascagni Ferdinando, Ugo Silva Dias, and Michel Daoud Yacoub

***Abstract*— This paper presents two closed-form approximations for the high-order statistics of the κ-μ Extreme fading model. The rationale for the approximations arose from the observation of the empirical behavior of the level crossing statistics for data whose first order statistics closely follow the κ-μ Extreme model. Field measurements are then used to validate the formulations.**

***Keywords*— κ-μ Extreme distribution, level crossing rate, average fade duration, field measurements, validation.**

***Resumo*— Este artigo apresenta duas aproximações em fórmula fechada para as estatísticas de ordem superior do modelo de desvanecimento κ-μ Extremo. A razão para as aproximações surge da observação das estatísticas empíricas de cruzamento de nível de dados nas quais as estatísticas de primeira ordem seguem de maneira muito próxima o modelo κ-μ Extreme. Medidas de campo são usadas para validar a formulação.**

***Palavras-Chave*— distribuição κ-μ Extrema, taxa de cruzamento de nível, tempo médio de desvanecimento, medidas de campo, validação.**

I. INTRODUCTION

IN wireless communications, a well investigated propagation phenomenon that increasingly and continuously raises the interest of the researchers is the fading. The occurrence of fading depends on several factors, including the environment and the propagation frequency. Its harshness ranges from very mild to extremely severe, and a number of models appear in the literature that reasonably well describe such a phenomenon in its various aspects. Among the fading distributions, Rayleigh, Hoyt, Weibull, Rice, and Nakagami-m, are the best known. Recently, more general fading models, namely α-μ [1], κ-μ [2], and η-μ [2], have been proposed that comprise the previous ones as special cases and that better fit experimental field data. Because wireless communications applications have been progressively diversified, not only outdoor and indoor environments have experienced their surge, but enclosed scenarios too. Enclosed and some indoor environments are characterized by very severe fading conditions. More specifically, due to the harsh variation of the received signal, a great deal of reception points may be found which are well below the receiver sensitivity. Therefore, even though the signal may not necessarily be nil, it may be sufficiently low with such an event occurring a sufficient number of times that the probability of finding it at such a condition may not be negligible. In addition, unlike the traditional environments, where different combinations of large number of multipath components lead to known fading channels, enclosed environments may present only a few radio paths, therefore rendering the utilization of the Central Limit Theorem inappropriate. Furthermore, known and useful propagation mechanisms (e.g., Plane Earth [3]) predict that direct and reflected waves may be combined to yield nulls at some reception points. In [4], the severe fading conditions – worse than that predicted by the Rayleigh case – in enclosed environments was named hyper-Rayleigh fading [4]–[6]. In [2], an extreme condition of fading was found for the κ-μ distribution. Such a condition, namely κ-μ Extreme, was later explored in [7]. In particular, field measurements conducted in a large transport helicopter, as reported in [4], and others collected in a university parking lot with moving cars and within a sports gymnasium were used to validate the κ-μ Extreme high-order statistics obtained here.

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Currently, the κ-μ Extreme model is limited to its first order statistics. Exploring high-order statistics is certainly of interest to fully characterize the fading channel. It is anticipated, however, that, specifically for the κ-μ Extreme case, this is a tough problem whose exact solution is still open. The aim of this paper is to find approximate expressions for the level crossing rate (LCR) and for the average fade duration (AFD) of this new channel.

The remainder of this work is structured as follows. Section II revisits the physical model and some expressions for the κ-μ Extreme fading. Section III develops the joint probability density function of the envelope and its time derivative. Section IV proposes two approaches to approximate the LCR and AFD statistics. Section V describes the conditions with which the field measurements have been conducted. Section VI compares the proposed approximations and the statistics obtained from field measurements. Section VII concludes the paper.

II. THE κ-μ EXTREME MODEL REVISITED

The κ-μ distribution is a general fading distribution that can be used to represent the small variation of the fading signal under LOS conditions. It includes as special cases important other distributions such as Rice (Nakagami-*n*) and Nakagami-*m* [2]. Therefore, One-Sided Gaussian and Rayleigh also constitute special cases of it. As its name implies, it is written in terms of two physical parameters, namely κ and μ. The parameter $\kappa > 0$ concerns the ratio between the total power of the dominant components and the total power of the scattered waves, whereas the parameter $\mu > 0$ is related to the multipath clustering. The κ-μ Extreme distribution was also originally proposed in [2] and arises as a particular case of κ-μ distribution, for which the fading parameters assume extreme values, i.e., $\kappa \to \infty$ (very strong LOS) and $\mu \to 0$ (very few multipaths), denoting very high severe fading conditions. According to [2], the PDF of the κ-μ Extreme envelope can be written in terms of the Nakagami-*m* fading parameter, m. Such result shows that for a given m, an infinite number of curves of κ-μ distribution can be obtained for appropriate values of κ and μ, rendering it well suited to field measurements in LOS conditions with very severe fading scenarios [2], [7].

For a fading signal with envelope R, with $\hat{r} = \sqrt{E(R^2)}$ being the *rms* value of R, the κ-μ Extreme PDF $f_{\mathrm{P}}(\rho)$ of the normalized envelope $\mathrm{P} = R/\hat{r}$ is given as [7]

$$f_{\mathrm{P}}(\rho) = \frac{4mI_1(4m\rho)}{\exp[2m(1+\rho^2)]} + \exp(-2m)\delta(\rho), \tag{1}$$

in which $I_\nu(\cdot)$ is the modified Bessel function of the first kind and order ν [8, Eq. 9.6.20], and $\delta(\cdot)$ is the Dirac delta function. For convenience, Equation (1) is now rewritten as

$$f_{\mathrm{P}}(\rho) = g(\rho) + \exp(-2m)\delta(\rho), \tag{2}$$

in which $g(\rho)$ is the continuous part of the PDF.

The corresponding CDF is given as [7]

$$F_{\mathrm{P}}(\rho) = 1 - Q_0\left(2\sqrt{m}, 2\sqrt{m}\rho\right), \tag{3}$$

in which $Q_0(\cdot,\cdot)$ is the *zero*-th Marcum Q-function [9].

III. Joint Distribution

For the LCR and AFD calculations, the knowledge of the joint PDF $f_{\mathrm{P},\dot{\mathrm{P}}}(\rho,\dot{\rho})$ of the normalized envelope P and its time derivative $\dot{\mathrm{P}}$ is required. As already mentioned, the κ-μ Extreme model constitutes a special case of the family of the κ-μ fading. In [10], Cotton and Scanlon showed that the envelope and its time derivative are independent random variables and that the latter is zero-mean Gaussian distributed with variance given by $\dot{\sigma}^2 = 2\pi^2\sigma^2 f^2$, in which f is the maximum Doppler shift in *Hz* and σ is the standard deviation of the Gaussian composing the κ-μ fading model. Therefore, the required joint PDF for the κ-μ Extreme case is given by

$$f_{\mathrm{P},\dot{\mathrm{P}}}(\rho,\dot{\rho}) = f_{\mathrm{P}}(\rho) \times f_{\dot{\mathrm{P}}}(\dot{\rho}), \tag{4}$$

where $f_{\dot{\mathrm{P}}}(\dot{\rho})$ can be written, using the appropriate random variable transformation $\dot{\mathrm{P}} = \dot{R}/\hat{r}$, as

$$f_{\dot{\mathrm{P}}}(\dot{\rho}) = \frac{\hat{r}}{\sqrt{2\pi}\dot{\sigma}} \exp\left(-\frac{\hat{r}^2\dot{\rho}^2}{2\dot{\sigma}^2}\right). \tag{5}$$

From [2], $\kappa = d^2/2\mu\sigma^2$ and $\hat{r}^2 = 2\mu\sigma^2 + d^2$, yielding

$$\sigma^2 = \frac{\hat{r}^2}{2\mu(1+\kappa)}. \tag{6}$$

By keeping the fading parameter $m = V(\mathrm{P})^{-1}$ constant, and κ and μ reaching their extreme values, i.e., infinity and zero, respectively, then $\kappa\mu = 2m$,

$$\sigma^2 = \frac{\hat{r}^2}{4m}, \tag{7}$$

$$\dot{\sigma}^2 = \frac{\pi^2 f^2 \hat{r}^2}{2m}. \tag{8}$$

Hence, using (8) in (5) results in

$$f_{\dot{\mathrm{P}}}(\dot{\rho}) = \frac{\sqrt{m}}{\pi^{3/2} f} \exp\left(-\frac{m\dot{\rho}^2}{\pi^2 f^2}\right). \tag{9}$$

IV. LCR and AFD

The classical way to obtain the LCR is by the Rice formula given by [11]

$$N_R = \int_0^\infty \dot{\rho} f_{\mathrm{P},\dot{\mathrm{P}}}(\rho,\dot{\rho}) d\dot{\rho}. \tag{10}$$

However, as pointed out by Rice himself, such a formulation can only be applied in case the joint PDF is continuous and the integral converges uniformly. The very characteristics of the κ-μ model shows that the first requirement is not fulfilled. The κ-μ Extreme envelope has a mixed probability distribution, i.e., it has both a continuous part and a discrete part. Therefore, (10) cannot be applied directly. One could try to circumvent this by using the formula of LCR derived for the κ-μ fading envelope and find its limit when the κ and μ parameters go to their extremes while keeping the signal power variance constant. Although, in this case, a formula can be attained, it does not lead to a physically plausible solution. Particularly, a trend towards an impulse at the origin is envisaged, meaning that the signal crosses this level an infinite number of times. However, at the vicinity of the zero-plus level the LCR is nil and increases with the increase of the level. Therefore, a new approach needs be found. Here we propose approximate formulations for LCR and AFD and maintain that an exact solution for this problem is a subject open for investigation. Two approximations are proposed. Such approximations arise from the observation of the level crossing statistics for signals for which the κ-μ Extreme distribution yields a good fit. Such an observation led to the following inference: above the level below which the signal plunges to give nulls at the reception, the level crossing statistics fit quite well with that one calculated using the continuous part of the distribution; below that value, as expected, such statistics remains approximately constant because of the sudden drop of the signal level. Bearing this in mind, the following closed-form approximations are proposed. All of them attempt to embed the probability mass at zero level within the continuous part of the PDF so as to have a final PDF that is no longer mixed.

A. Approximation A

Approximation A is given as

$$f_{\mathrm{P}}(\rho)_{cont.} = \begin{cases} g(\rho_0 - \rho) + g(\rho), & 0 \le \rho \le \rho_0 \\ g(\rho), & \rho > \rho_0 \end{cases}, \tag{11}$$

in which ρ_0 must be obtained such that (11) be a PDF such as

$$\int_0^{\rho_0} g(\rho) d\rho = \exp(-2m), \tag{12}$$

which is, indeed, the probability at $\rho = 0$. Equation (12) can be solved now to yield

$$Q_0\left(2\sqrt{m}, 2\sqrt{m}\rho_0\right) = 1 - 2\exp(-2m). \tag{13}$$

Finally, the closed-form LCR for the first proposed κ-μ Extreme continuous approximation is attained using (4) and (11) in (10) as

$$N_R(\rho) = \begin{cases} 0.5f\sqrt{\frac{\pi}{m}}\left[g(\rho_0 - \rho) + g(\rho)\right], & 0 \le \rho \le \rho_0 \\ 0.5f\sqrt{\frac{\pi}{m}}g(\rho), & \rho > \rho_0 \end{cases}. \tag{14}$$

B. Approximation B

Approximation B is given as

$$f_{\mathrm{P}}(\rho)_{cont.} = \begin{cases} g(\rho_0), & 0 \le \rho \le \rho_0 \\ g(\rho), & \rho > \rho_0 \end{cases}, \tag{15}$$

so that,

$$\int_{\rho_0}^{\infty} g(\rho)d\rho + g(\rho_0)\rho_0 = 1. \tag{16}$$

As before,

$$Q_0\left(2\sqrt{m}, 2\sqrt{m}\rho_0\right) + g(\rho_0)\rho_0 = 1. \tag{17}$$

Then,

$$N_R(\rho) = \begin{cases} 0.5f\sqrt{\frac{\pi}{m}}g(\rho_0), & 0 \le \rho \le \rho_0 \\ 0.5f\sqrt{\frac{\pi}{m}}g(\rho), & \rho > \rho_0 \end{cases}. \tag{18}$$

Both Equations (13) and (17) must be solved for ρ_0. To this end, built-in routines available in classical computing softwares (e.g. *MATHEMATICA*) can be used in an efficient and straightforward manner.

Finally, the AFD is obtained for both Approximations as,

$$T_R(\rho) = \frac{F_{\mathrm{P}}(\rho)}{N_R(\rho)} = \frac{1 - Q_0\left(2\sqrt{m}, 2\sqrt{m}\rho\right)}{N_R(\rho)}. \tag{19}$$

V. FIELD MEASUREMENTS

The aim to obtain field data is to compare the two theoretical approaches for the high-order statistics of the κ-μ Extreme distribution against the empirical statistics. To this end, two scenarios were explored: (i) a parking lot with cars aligned and moving vehicles; and (ii) a large and nearly empty gymnasium. In (i), both transmitter and receiver were placed below the height of the cars and a LOS condition was always in place. In (ii), the condition was similar to that of (i), except for the cars, with some people walking by. The receiver equipment setup consisted of a vertically polarized omnidirectional receiving antenna, a low noise amplifier, a spectrum analyzer, data acquisition apparatus, a notebook computer, and a distance transducer for carrying out the signal at a $\lambda/14$ sampling [12]–[14]. The transmission consisted of a CW tone at 1.8 GHz. The spectrum analyzer was set to zero span and centered at the desired frequency, and its video output used as the input of the data-acquisition and processing equipment. The local mean was estimated by the moving average method, with optimum windows length from 45λ to 60λ [15], equivalent to 630 and 840 measured envelope samples, respectively. From the collected data, the long term fading was filtered out, then the fading parameter m could be estimated.

Fig. 1 illustrates the κ-μ Extreme process, where the severe variation of the received signal with deep fades can be observed.

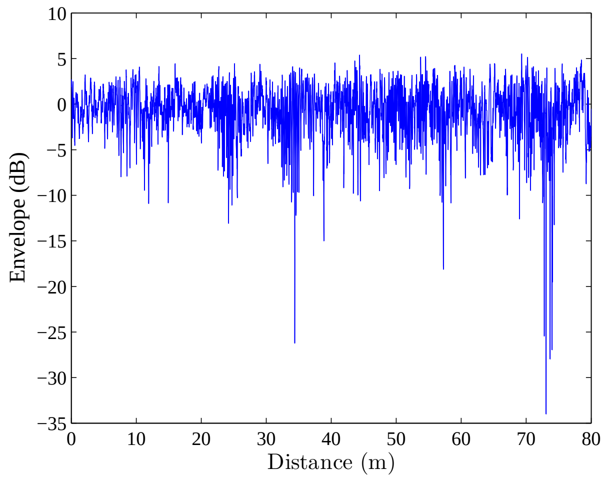

Fig. 1. Illustration of the κ-μ Extreme process ($m = 3.53$) in a parking lot for a 80m run.

TABLE I
ESTIMATED PARAMETERS FOR THE APPROXIMATIONS

Data	***m***	ρ_0	
		(A)	**(B)**
data #1	3.25	0.143	0.130
data #2	3.98	0.116	0.105

VI. RESULTS

In order to compare the curve fitting of the empirical high-order statistics (from field measurements) and the proposed approximations of the κ-μ Extreme high-order statistics, the maximum Doppler shift parameter f has been adjusted to give the curves the best fit. Figs. 2 to 9 compare approximate expressions (continuous curve) with empirical curve (circles) for both Approximations A and B and two fading datas, named #1 for the parking lot scenario, and #2 for the gymnasium scenario. It is noted that the two proposed approximations yield almost the same curve shapes, the difference between them appearing at very low levels, i.e. near $\rho = 0$, since ρ_0 typically assumes relatively low values, as can be seen in Table I. Another point to be observed is that, in practice, the level crossing rate at threshold $\rho = 0$ does not make sense. However, the κ-μ Extreme distribution predicts a non-nil probability of signals nulls, such that the LCR at the origin may be interpreted as the arriving or departure rate of the signal at the threshold zero. And the AFD relates to the time the signal remains at that level. Table II shows the values found for the empirical and theoretical high-order statistics at $\rho = 0$.

In a general, both approximations yield an excellent fit as compared to the empirical high-order statistics, including their values at $\rho = 0$.

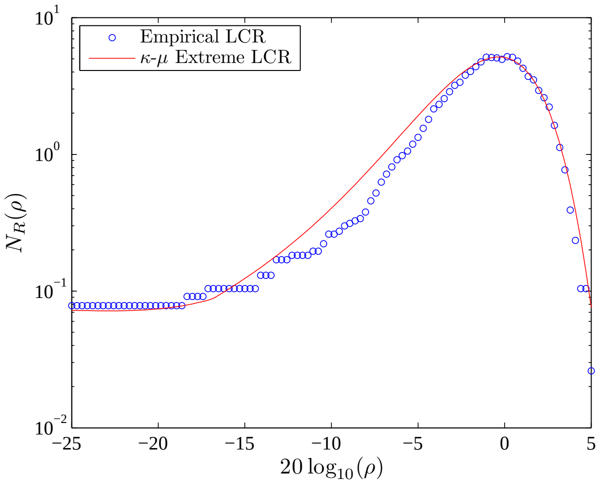

Fig. 2. Comparison of LCR with respect to data #1 and approach A ($m = 3.25$, $f = 7.45$ Hz).

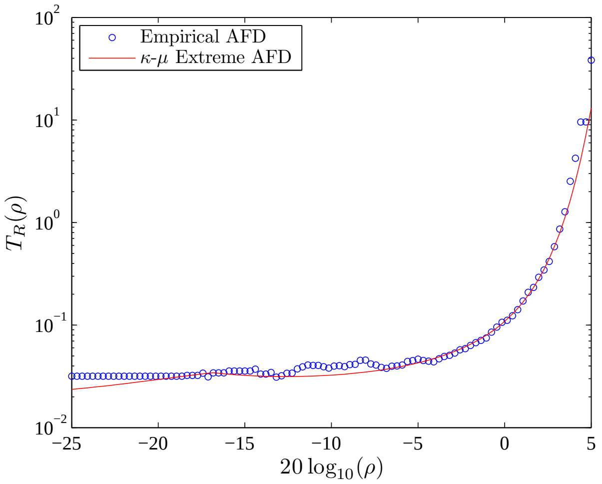

Fig. 3. Comparison of AFD with respect to data #1 and approach A ($m = 3.25$, $f = 7.45$ Hz).

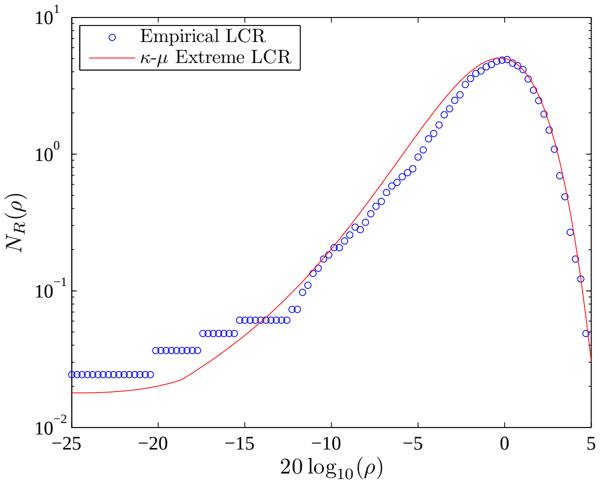

Fig. 4. Comparison of LCR with respect to data #2 and approach A ($m = 3.98$, $f = 7.25$ Hz).

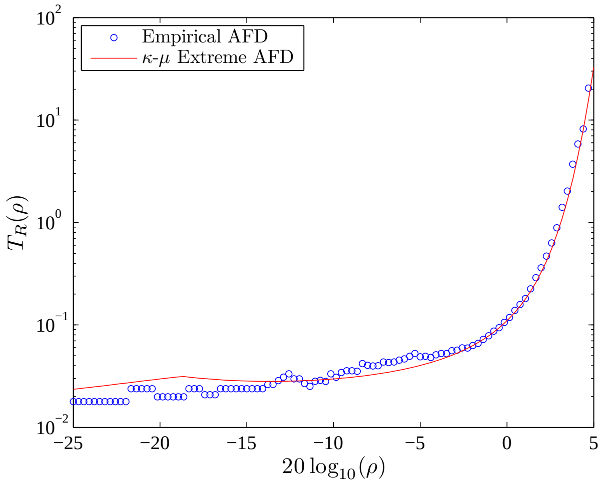

Fig. 5. Comparison of AFD with respect to data #2 and approach A ($m = 3.98$, $f = 7.25$ Hz).

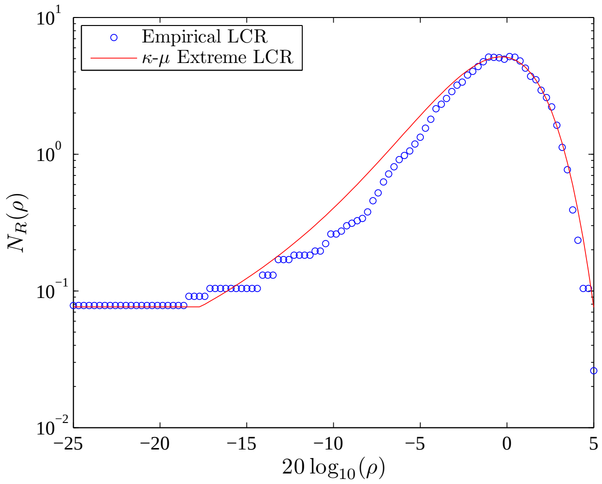

Fig. 6. Comparison of LCR with respect to data #1 and approach B ($m = 3.25$, $f = 7.45$ Hz).

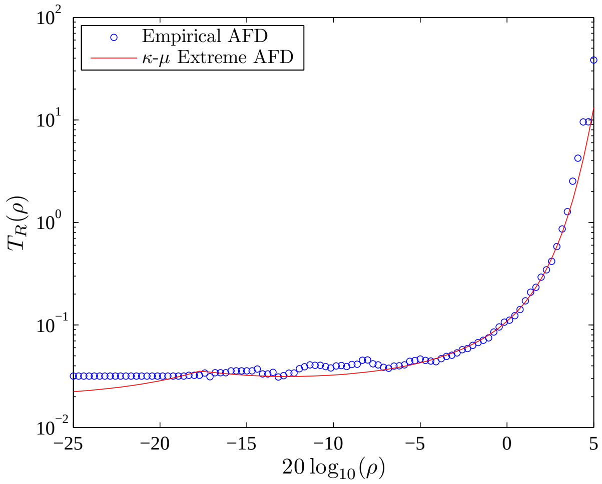

Fig. 7. Comparison of AFD with respect to data #1 and approach B ($m = 3.25$, $f = 7.45$ Hz).

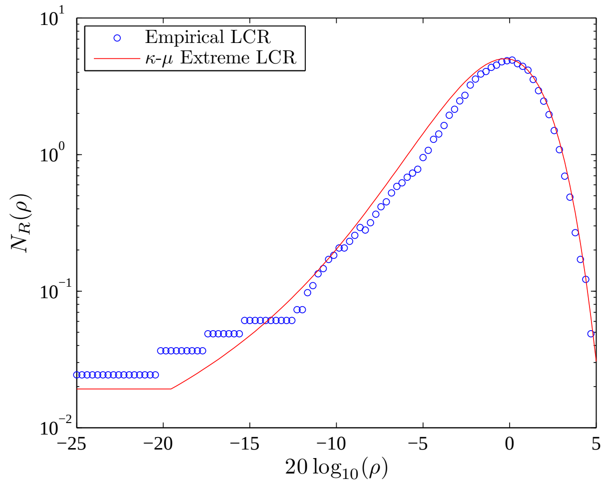

Fig. 8. Comparison of LCR with respect to data #2 and approach B ($m = 3.98$, $f = 7.25$ Hz).

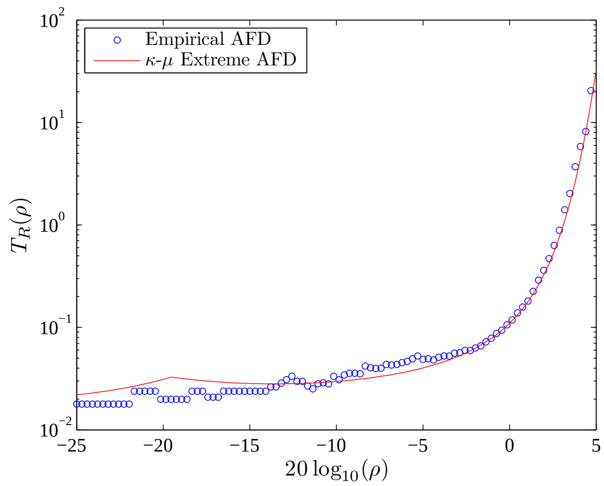

Fig. 9. Comparison of AFD with respect to data #2 and approach B ($m = 3.98$, $f = 7.25$ Hz).

TABLE II

EMPIRICAL AND THEORETICAL LCR AND AFD VALUES AT THE ORIGIN

Data	**LCR at** $\rho = 0$			**AFD at** $\rho = 0$		
	Emp.	**(A)**	**(B)**	**Emp.**	**(A)**	**(B)**
data #1	0.078	0.087	0.076	0.031	0.017	0.019
data #2	0.024	0.022	0.019	0.011	0.015	0.018

VII. CONCLUSIONS

This paper presented two closed-form approximations for the high-order statistics of the κ-μ Extreme distribution, thus circumventing the limitation of the Rice formula, which applies only to continuous processes. The approximations were compared with empirical curves from some field measurements and showed excellent adherence.

Future works include finding new approximations as well as defining means to arrive at the exact solution.

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