The Use of Discrete Prolate Spheroidal Sequences and Trig Prolates to Compressed Sensing

Juliana M. de Assis and Edmar C. Gurjão

***Abstract*— Compressed sensing may offer the possibility to acquire certain signals at a rate below Nyquist with guaranteed perfect recovery of these signals. In the present article, we investigate the utilization of Discrete Prolate Spheroidal Sequences as a sparsifying basis for multiband signals, from where compressed sensing may be applied. We also compare their use with the use of trig prolates, a similar and simpler to compute basis, in the context of compressed sensing. Using CoSaMP as the reconstruction algorithm we demonstrate the infeasibility of trig prolates as a basis for perfect recovery.**

***Keywords*— Discrete Prolate Spheroidal Sequences, Trig Prolates, Compressed Sensing, Compressive Sampling Matching Pursuit.**

I. INTRODUCTION

Since the works of Nyquist and Shannon related to analog to digital conversion, technology evolved in a frenetic way. Nyquist theorem establishes rules to sampling and recovering analog signals as long as the sampling rate is greater than or equal to twice its greater frequency component, $B_{nyq} \geq 2B$ [1]. Development of algorithms and hardware permitted this theorem to be used extensively, as also allowed the observation of Moore's Law [2]. Nevertheless, the acquisition of signals with ever growing bandwidth turns hardware more expensive, imposing restrictions to digital signals processing.

In this context, the theory of compressed sensing, or compressive sampling, CS, has been developed by Donoho, who searched for a way to capture and transmit only the information present in a signal [3], and by Candès and Tao, who simultaneously tried to accomplish this by random projections [4]. If analog signals can be approximately sparse in some basis, as will be latter explained, then it is possible to use much less than the Nyquist number of samples to acquire it. This paper investigates the use of Discrete Prolate Spheroidal Sequences (DPSS) as basis in CS, as also the use of trig prolates as basis substitutes for DPSS. DPSS are most indicated when dealing with multiband signals, that is, signals whose Fourier transform is concentrated on a small number of continuous intervals or bands [5]. These are different from multitone signals, which are frequently used in the context of telecommunications. Multitone signals are well expressed by the Discrete Fourier Transform (DFT) simply.

This article is organized as follows, in Section II fundamentals of CS are presented, in Section III the DPSS are described, as also their use as a basis. Section IV brings the results of simulations with the use of different basis (DFT, DPSS, and trig prolates) and the same reconstruction algorithm - CoSaMP. In Section V we review the application of DPSS in CS. Finally, in Section VI conclusion and perspective for future works are presented.

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II. COMPRESSED SENSING BACKGROUND

In this section some mathematical concepts are presented to help the understanding of CS. A signal $x \in \mathbb{R}^N$ is S-sparse if

$$|supp(x)| = ||x||_0 \leq S. \tag{1}$$

The $|supp(x)|$ is also called the ℓ_0-norm, though it is not a true norm, and it counts the number of nonzero components.

The ℓ_p-norm, for $1 \leq p < \infty$, is given by:

$$||x||_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} . \tag{2}$$

If x is not sparse, it might be written in a basis Ψ, where it has a sparse representation. Examples of basis are spikes, B-splines, wavelets, among other possibilities. Signal x in its new representation becomes sparse or compressible α.

$$x = \Psi\alpha. \tag{3}$$

Given a N dimensional S sparse signal x, and a measurement matrix $A_{M\times N}$, it is possible to maintain the information from the S components of α, by $y = A\Psi\alpha = Ax$. In CS, to use a sub-Nyquist rate, $M < N$.

Coherence between two pairs of orthobases is defined (the restriction to pairs of orthobases is not essential [6]):

$$\mu(A, \Psi) = \sqrt{N} \max_{1\leq k,j\leq N} |\langle A_k \psi_j \rangle|. \tag{4}$$

Random matrices are largely incoherent with any fixed basis [6]. Incoherence between measurement matrix A and basis Ψ is a sufficient condition in CS [7].

There are many properties concerning the design of measurement matrices, one of these is that the minimum number M of measures must obey $M \geq 2S$, where S is the signal sparsity. The reason is that sensing matrices should project every two S-sparse vectors in distinct samples, thus every set of $2S$ columns of A must be nonsingular [8]. However, it is more reliable that CS will function properly if the measurement matrix follows the restricted isometry property (RIP). Matrix $A\Psi$ satisfy RIP of order S if there exists one constant $\delta_S \in (0, 1)$ such that

$$\sqrt{1-\delta_S} \leq \frac{||A\Psi\alpha||_2}{||\alpha||_2} \leq \sqrt{1+\delta_S} \qquad (5)$$

for every α such that $||\alpha||_0 \leq S$, that is, $A\Psi$ preserves the norm of S-sparse vectors. Furthermore, if $||\alpha||_0 = ||\alpha'||_0 = S$, $||\alpha - \alpha'||_0 \leq 2S$, and $A\Psi$ satisfy RIP of order $2S$, it also preserves the euclidean distance between S-sparse vectors [5].

On the other hand, matrix A satisfy Ψ-RIP if there is a constant $\delta_S \in (0,1)$ such that

$$\sqrt{1-\delta_S} \leq \frac{||A\Psi\alpha||_2}{||\Psi\alpha||_2} \leq \sqrt{1+\delta_S} \qquad (6)$$

for every α such that $||\alpha||_0 \leq S$. When Ψ forms an orthonormal basis, the properties of (5) and (6) are equivalent. However, RIP and Ψ-RIP are slightly different concepts. In the former, the norm preservation is guaranteed for all the sparse vectors α, while in the latter the norm preservation is guaranteed for all signals with a sparse representation $x = \Psi\alpha$ [5].

Many types of random matrices follow RIP, with few measurements M. Unfortunately, it is an open problem whether deterministic sampling matrices also follow RIP [8], [9], [10]. Measurement matrices $A_{M\times N}$ are random, chosen with independent and identically distributed entrances such that: $\mathbb{E}(A[m,n]) = 0$ and $\mathbb{E}(A[m,n]^2) = 1/M$, from subgaussian distributions, such as Gaussian, Rademacher and uniform. In those cases, $||Ax||_2^2$ is concentrated close to $||x||_2^2$ and if M is greater than certain value, A follows RIP of order S (with probability depending on a parameter). An analogous result is found for matrices following the Ψ-RIP of order S [5].

There are many reconstruction algorithms for recovering one signal that has passed through CS, but they are not trivial since the matrix $A_{M\times N}$, when $M < N$, constitutes a system with more unknowns than equations. The original problem is related to the use of ℓ_0-norm and is a nonconvex minimization problem, with NP complexity. One alternative choice to solve is shown in (7), called the ℓ_1-minimization problem or basis pursuit [7], [6], [11]. The ℓ_1-norm is a measure that when used in optimization problems tends to find solution over the axes, which is equivalent to finding sparse solutions [12], [13]. This characteristic is well-suited for the problem in hand.

$$\min_{\tilde{x}\in\mathbb{R}^N} ||\tilde{x}||_1 \text{ subject to } A\Psi\tilde{x} = y. \qquad (7)$$

Generically, recovery algorithms can be classified as:

- Greedy algorithm: recovers the signal each step by making locally optimal choices;
- Convex relaxation algorithms: solve convex problems;
- Combinational algorithms: rapid reconstruction through group testing.

CoSaMP (compressive sampling matching pursuit) is a greedy algorithm that utilizes ideas of combinational algorithms to increase its speed [8]. CoSaMP was chosen for recovering signals in this work because of its availability in the toolbox from reference [5].

III. Discrete Prolate Spheroidal Sequences

In this section we present Discrete Prolate Spheroidal Sequences - DPSS and how they can form a sparsifying basis. DPSS are useful for projecting digital filters, also in pulse shaping, secure communications and in phase amplitude modulation (PAM) [14].

Consider the amplitude spectrum $X(f) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi fn}$ associated with a complex sequence x_n. It is possible to associate an energy value to an indexed sequence limit:

$$E(n_1,n_2) = \sum_{n=n_1}^{n_2} |x_n|^2, \qquad (8)$$

as also to associate this to spectral amplitude:

$$x_n = \int_{-1/2}^{1/2} X(f)e^{-j2\pi fn}df, \quad n = 0, \pm1, \ldots. \qquad (9)$$

According to Parseval's theorem, one can also write:

$$E = \sum_{n=-\infty}^{\infty} |x_n|^2 = \int_{-1/2}^{1/2} |X(f)|^2 df. \qquad (10)$$

It is said that x_n is bandlimited with bandwidth W if the spectral amplitude of the sequence x_n vanishes for $W < |f| \leq 1/2$, where $W < 1/2$ is a positive number. Analogously, given two integers $n_1 \leq n_2$, the sequence x_n is time-limited if it vanishes when $n < n_1$ or $n > n_2$.

Except for the trivial all zero sequence, it is known that it is impossible to have sequences both time- and bandlimited, simultaneously. It is plausible to ask, however, how near to time-limited can a bandlimited sequence be, or what is its maximum possible concentration:

$$\lambda = \frac{E(N_0, N_0+N-1)}{E(-\infty,\infty)} = \frac{\sum_{n=N_0}^{N_0+N-1} |x_n|^2}{\sum_{n=-\infty}^{\infty} |x_n|^2}, \qquad (11)$$

among all the other sequences bandlimited to W. In this sense, the most concentrated sequence is proportional to a DPSS with parameters N and W [15].

For each $l = 0, 1, \ldots, N-1$, the DPSS $\{v_n^{(l)}(N,W)\}$ are defined as the real solution for the system of equations:

$$\sum_{m=0}^{N-1} \frac{\sin 2\pi W(n-m)}{\pi(n-m)} v_m^{(l)}(N,W) = \lambda_l(N,W) v_n^{(l)}(N,W), \quad n = 0, \pm1, \pm2, \ldots$$

normalized such that

$$\sum_{m=0}^{N-1} v_m^{(l)}(N,W)^2 = 1,$$

$$\sum_{m=0}^{N-1} v_m^{(l)}(N,W) \geq 0,$$

$$\sum_{m=0}^{N-1} (N-1-2m) v_m^{(l)}(N,W) \geq 0.$$

The $0 < \lambda_{N-1}(N,W) < \cdots < \lambda_1(N,W) < \lambda_0(N,W) < 1$ are the ordered eigenvalues of a differential equation that is related to the prolate spheroidal wave functions [15].

A. DPSS Basis

Davenport [5] presents concatenated basis DPSS, each modulated for center F_i of certain frequency band Δ_i, so as to write:

$$F_i = -\frac{B_{nyq}}{2} + \left(i + \frac{1}{2}\right) B_{band}, \quad i \in \{0, 1, \ldots, J-1\}, \quad (12)$$

where $J = B_{nyq}/B_{band}$, $W = B_{band}T_s/2$, $T_s \leq 1/B_{nyq}$, and for each i, with $f_i = F_i T_s$:

$$\Psi_i = [E_{f_i} S_{N,W}]_k, \quad (13)$$

where k indicates that the first k columns of the matrix were used. E_{f_i} is the $N \times N$ matrix given by:

$$E_{f_i}[m,n] = e^{j2\pi f_i m}, \text{ if } m = n,$$

and $E_{f_i}[m,n] = 0$, otherwise. $S_{N,W}$ is $N \times N$ matrix with N DPSS vectors (constructed with parameters N and W) as columns.

The multiband DPSS basis is written concatenating all Ψ_i:

$$\Psi = [\Psi_0 \Psi_1 \ \ldots \ \Psi_{J-1}]. \quad (14)$$

Matrix Ψ has size $N \times kJ$.

IV. Simulations

This subsection compares the performance of CS when using DPSS, DFT ($N \times N$) or trig prolates as basis. The reconstruction algorithm is CoSaMP, using the software *Matlab* and the toolbox of reference [5]. First simulation compares the use of the DFT basis and the DPSS basis. Sampling time is supposedly $T_s = \frac{1}{B_{nyq}}$ and the number of possible bands is $J = \frac{B_{nyq}}{B_{band}} = 256$. For each k value considered, $D = kJ$ and $\Psi_{N \times D}$ is the basis defined in (14). The half digital bandwidth parameter is set to $W = \frac{B_{band}T_s}{2} = \frac{1}{512}$ and there will be considered sample vectors of size $N = 1024$, so that $2NW = 4$. The multiband signals are generated from $K = 5$ random bands, from J possibilities, and made by adding 50 exponentials within each band of random frequencies, not aligned with the "*Nyquist grid*". The results are given by the signal to noise ratio (SNR):

$$\text{SNR} = 20 \log_{10} \frac{||x||_2}{||x - \hat{x}||_2} \text{dB},$$

where x is the original signal, $\hat{x}$ is the reconstructed signal by the algorithm CoSaMP. The measurement matrix was Gaussian, with dimensions $M \times N$, where $M = N/2$.

Figure 1 shows the mean SNR over 50 trials when the number of columns of each Ψ_i modulated DPSS basis is varied. It is visible that reconstruction using DFT basis stays almost the same when this number varies, as expected (because we did not change the DFT basis). However, it is clear from simulations that when Ψ is an overcomplete basis, that is, when each Ψ_i is constructed using more than $2NW$ DPSS columns as vectors, there are far better results in reconstruction.

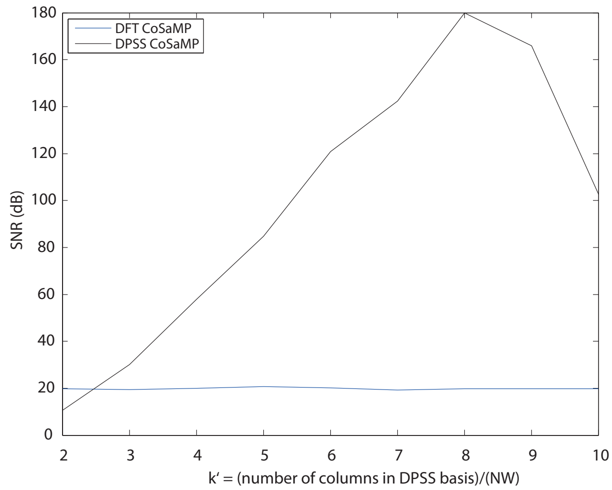


Fig. 1. SNR comparison between reconstruction using DFT and DPSS basis. The parameter k' of the number NWk' of columns used in the basis was varied and the resultant mean SNR over 50 trials was calculated.

A second simulation compares the use of the DPSS with small number of vectors as basis, the use of DPSS as an overcomplete basis and trig prolates as basis. The ability to accurate and rapidly compute DPSS is important specially for implementation of multitaper approach to spectral estimation. Some of the methods to generate DPSS are: from the defining equation, from inverse iteration, from numerical integration and from tridiagonal formulation. There are also substitutes for DPSS, which are more easily computed, when $NW = 2$, 3, or 4, called trig prolates [16]. However, trig prolates are not suited for applications with CS, at least when using the CoSaMP algorithm for reconstruction, since the easily computed ones actually give a small number of column vectors (maximum $NW + 1$ columns), and thus cannot compose the overcomplete basis that enables better performance.

We compared the performance of DPSS basis, with only $k = NW + 1$ columns, and the basis using trig prolates, when $NW = 4$, $N = 1024$. Simulations were made with parameters similar to those of previous simulation, with multiband signals made by adding 50 exponentials in each of $K = 5$ active bands from $J = 128$ possible bands and measurement matrix was also Gaussian ($N/2 \times N$). Figure 2 shows one example where recovered signals were very similar - which was a consistent result - but they were not very similar to the original one. The mean SNR over 50 trials when using the DPSS basis with column size $NW + 1 = 5$ was 4.09dB, and when using trig prolates it was 4.06dB, practically the same. When using the same parameters but with an overcomplete basis, with column size $8NW$, the mean SNR rises to about 205dB, which is a much better result. Another example shows the comparison between original signal and the recovery signals when the three different bases were used, in Figure 3.

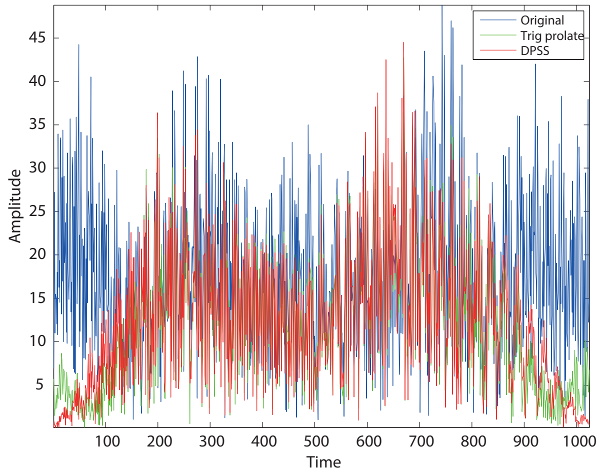
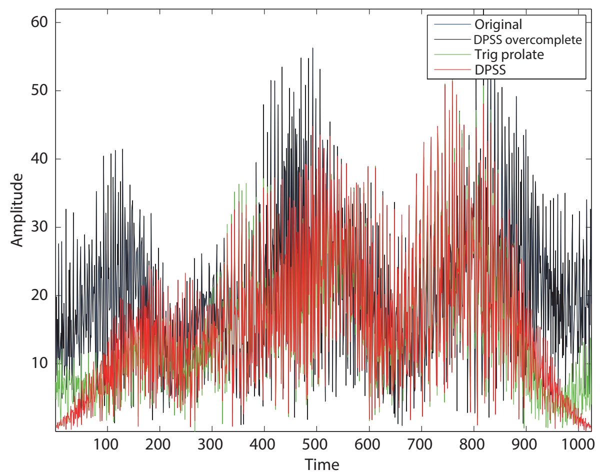
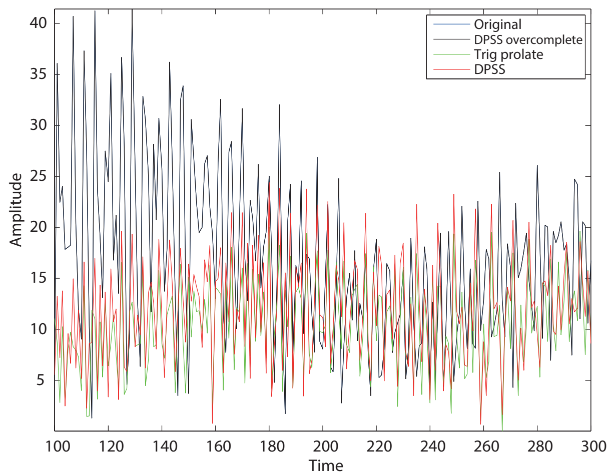


Fig. 2. One example comparison between original signal and reconstructed signals, when trig prolates or DPSS bases were used, in time domain.



(a) Signals in total duration



(b) Zoomed signals

Fig. 3. One example comparison between original signal and reconstructed signal, when trig prolates, DPSS or DPSS overcomplete bases were used. The blue curve of the original signal is almost totally covered by the reconstructed signal when DPSS overcomplete basis is used (black curve).

V. Applications

There has been application of CS jointly with DPSS not only in telecommunications as also in medicine. Reference [17] applies the modulated DPSS to accelerometry signal, which are useful in medicine. Dysphagic patients (i.e., patients suffering from swallowing difficulty) usually deviate from the well-defined pattern of healthy swallowing. Swallowing accelerometry employs an accelerometer as a sensor during cervical auscultation. CS is necessary since there is a large volume of redundant samples by continuous monitoring. The use of modulated DPSS, there is, of a basis $M_l(N, W, F_m; n) = e^{j2\pi F_m n} v_n^{(l)}(N, W)$, where $v^{(l)}(N, W)$ is a DPSS, resulted in accurate representations of the signals with about 50% of Nyquist samples.

In telecommunications, there is the development of a wideband compressive radio receiver (WCRR) architecture and algorithmic approach, for a WCRR that performs processing such as filtering and detection entirely in the compressive domain. Specifically, there is one of the radio's component that involves the interference calculation and filtering, with a projection matrix computed such that the spectral components of the interfering signal lie in its nullspace [18]. The model interference is made with DPSS. There are also applications for a project of radars using the compressive approach, where the procedure for nulling the interfering signal involves again the computation of DPSS [19]. Finally, there is basis expansion models using DPSS, with specific combination of DPSS and DFT basis functions, which yields functions that are still effectively zero outside an index range but, within that interval, preserve the sparsity obtained with the DFT basis [20], in the context of the application of CS to the estimation of doubly selective channels within pulse-shaping of multicarrier systems. The combination of DPSS and DFT bases yields superior performance to the DFT pure basis alone, especially at high SNR.

VI. Conclusion

We have studied and tested the use of a new basis, that utilizes DPSS, to sparsely represent multiband signals. We have compared the recovery of signals that passed through CS using CoSaMP algorithm, with both DFT and modulated DPSS bases, where the latter has improved performance over the former when the latter is overcomplete. We have tested trig prolates as basis and noted that they are not suited for substituting overcomplete DPSS basis, in the context of CS, with CoSaMP reconstruction. There is application of DPSS basis in medicine, and joint CS and DPSS applications are present in telecommunications, where we can cite radar and radio's architecture, as also in multicarrier systems. Despite all the cited applications, the sophisticated methods to generate DPSS leave this great basis still underused for CS implementation (and, unfortunately, the simpler trig prolates do not help either).

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